3. KRUTIKOV V. S., *One-dimensional Problems of the Mechanics of a Continuous Medium with Moving Boundaries.* Naukova Dumka, Kiev, 1985.
4. KRUTIKOV V. S., On the determination of the pressure on the moving boundary of a plasma piston. *Pis'ma v KhTF* **14**, 6, 510–514, 1988.
5. KRUTIKOV V. S., Approximate evaluation of the effect of the penetrability of the moving boundary of a plasma piston. *Pis'ma v ZhTF* **15**, 14, 45–48, 1989.
6. LANDAU L. D. and LIFSHITS E. M., *Mechanics of Continuous Media*. Gostekhizdat, Moscow, 1954.
7. VINOGRADOV I. M. (Ed.), *Mathematical Encyclopaedia* (Sovetskaya Entsiklopedia) Vol. 2. Moscow, 1979.
8. SLEPYAN L. I., On the equations of the dynamics of an axially symmetric cavity in an ideal compressible liquid. *Dokl. Akad. Nauk SSSR* **282**, 4, 809–813, 1985.
9. ROZHDESTVENSKII B. L. and YANENKO N. N., *Systems of Quasilinear Equations and their Applications to Gas Dynamics*. Nauka, Moscow, 1978.

Translated by D.L.

J. Appl. Maths Mechs Vol. 55, No. 6, pp. 953–957, 1991
Printed in Great Britain.
0021-8928/91 $15.00+.00

A NON-AXISYMMETRIC CONTACT PROBLEM IN THE CASE OF A NORMAL LOAD APPLIED OUTSIDE THE AREA OF CONTACT†

V. I. MOSSAKOVSKI and YE. V. POSHIVALOVA

Dnepropetrovsk

(*Received* 11 *July* 1990)

1. A FORMULA describing the effect of a load acting outside a circular stamp in a plane is known [1]. Below we propose a novel approach to the study of the pressure under a non-axisymmetric plane stamp when normal forces are applied to the free surface of an elastic half-space. The approach includes the method, proposed by Mossakovskii, of reducing the three-dimensional problem of potential theory to a plane problem. The main merit of this method, as compared with that in [2] based on the Sommerfeld method, is the possibility of constructing effective numerical algorithms, since any subsequent approximation can be constructed indepndently of the preceding one, by adding some supplementary terms. The problem in question is reduced, in the final analysis, to a system of plane problems of potential theory whose boundary conditions contain trigonometric polynomials with unknown coefficients, which can be determined from the condition that the solution is regular within the area of contact.

Let a normal force R be applied to the surface of an elastic half-space outside the area of contact at the point ξ, η. As a result, additional pressure and normal displacement occur under the stamp.

We will assume that the normal displacement of the stamp $W(\rho, \alpha)$ is identical with the displacement of the

†*Prikl. Mat. Mekh.* Vol. 55, No. 6, pp. 1062–1065, 1991.

boundary of the half-space caused by the action of the force R. Then, in order to determine the pressure under the stamp we must determine the value of the function $P(\rho, \alpha) = \varphi z'(\rho, \alpha)$ in the region S, and the function $\varphi(\rho, \alpha, z)$, harmonic in the half-space $z \leqslant 0$, is given by the boundary conditions

$$\varphi(x, y, 0) = -R\frac{((x-\xi)^2+(y-\eta)^2)^{1/2}}{2(1-\nu^2)}, \quad |\rho|<\rho(\alpha)$$
$$\varphi_z'(x, y, 0) = 0, \quad |\rho|>\rho(\alpha) \tag{1.1}$$

where $\rho(\alpha)$ is the equation of the boundary of the area of contact and ρ, α are polar coordinates.

Changing to new variables $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $\eta = c\sin\psi$, $\xi = c\cos\psi$, $\alpha = \varphi - \psi$ and expanding the numerator in (1.1) in a Fourier series, we obtain

$$c^{-1}(1-\rho c^{-1}e^{-i\alpha})^{-1/2}(1-\rho c^{-1}e^{i\alpha})^{-1/2} = c^{-1}\{f_0(\rho) + 2(f_1(\rho)\cos\alpha + f_2(\rho)\cos 2\alpha + \ldots)\}$$
$$f_0(\rho) = a^2(1) + a^2(2)\rho^2c^{-2} + a^2(3)\rho^4c^{-4} + \ldots$$
$$f_1(\rho) = a(1)\,a(2)\,\rho c^{-1} + a(2)\,a(3)\,\rho^3c^{-3} + \ldots$$
$$a(1) = 1, \quad a(2) = \frac{1}{2}, \quad a(3) = \frac{1\cdot 3}{2\cdot 4}, \ldots$$

The function $\varphi(\rho, z, \alpha)$ can also be expanded in series

$$\varphi(\rho, z, \alpha) = \varphi_0(\rho, z) + \varphi_1(\rho, z)\cos\alpha + \varphi_2(\rho, z)\cos 2\alpha + \ldots$$

The functions $\varphi_n(\rho, 0)$, $\varphi_{nz}(\rho, 0)$ can be written in the form of contour integrals [3]. Then the boundary conditions (1.1) will become

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_n(s)\,2^{1-s}\Gamma\left(\frac{n}{2}-\frac{s}{2}\right)\Gamma^{-1}\left(1+\frac{n}{2}+\frac{s}{2}\right)^{s-2}ds = 0, \quad |\rho|>\rho(\alpha)$$
$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_n(s)\,2^{2-s}\Gamma\left(\frac{1}{2}+\frac{n}{2}-\frac{s}{2}\right)\Gamma^{-1}\left(\frac{1}{2}+\frac{n}{2}+\frac{s}{2}\right)\rho^{s-1}\,ds = -R\,\frac{f_n(\rho, \alpha)}{2(1-\nu^2)} \tag{1.2}$$
$$|\rho|<\rho(\alpha), \quad n = 0, 1, 2, \ldots$$

2. The non-axisymmetric problem (1.2) can be reduced, for every value of α, in the case of even n, to the axisymmetric problem with boundary conditions [3]

$$\Phi(\rho, 0, \alpha) = \frac{R}{2(1-\nu^2)}\Bigg(\sum_{k=0}^{\infty}\left(\frac{\rho}{c}\right)^{2k}a^2(k+1) + 2\sum_{k=1}^{\infty}\cos 2k\alpha \times$$
$$\times\sum_{n=1}^{\infty}\left(\frac{\rho}{c}\right)^{2n}a^2(n+k) + G(\rho, \alpha)\Bigg), \quad |\rho|<\rho(\alpha) \tag{2.1}$$
$$\Phi_z'(\rho, 0, \alpha) = 0, \quad |\rho|>\rho(\alpha)$$
$$G(\rho, \alpha) = G_0 + G_2\rho^2 + G_4\rho^4 + \ldots, \quad G_0(\alpha) = \alpha_0^2\cos 2\alpha +$$
$$+\,\alpha_0^4\cos 4\alpha + \ldots, \quad G_2(\alpha) = \alpha_2^4\cos 4\alpha + \alpha_2^6\cos 6\alpha + \ldots, \ldots$$

The pressure under the stamp is connected with the axisymmetric function by the relation

$$P(\rho, \alpha) = \Phi_z'(\rho, 0, \alpha) \tag{2.2}$$

Problem (2.1) can, in turn, be reduced to the potential problem relative to some function $Q(\xi, \eta)$, harmonic in the auxilliary plane $\eta \leqslant 0$ and antisymmetric with respect to ξ. The boundary conditions of this problem will have the form

$$Q_\eta'(\xi, 0) = 0, \; |\xi| > \rho(\alpha)$$

$$Q_\xi'(\xi, 0) = \frac{R}{2(1-\nu^2)}\left(\sum_{k=1}^{\infty}\left(\frac{\xi}{c}\right)^{2k} + 2\sum_{k=1}^{\infty}\cos(2k\alpha)\sum_{n=k}^{\infty}\left(\frac{\xi}{c}\right)^{2n} + G(\xi, \alpha)\right), \; |\xi| < \rho(\alpha), \tag{2.3}$$

The functions $Q(\xi, \eta)$ and $\Phi(\xi, \eta)$ are connected by the relation

$$Q_\xi'(\xi, 0) = \Phi_z'(\xi, 0)\,\xi \tag{2.4}$$

3. Just as in the case of even n, the non-axisymmetric problem (1.2) can be reduced, for every value of α, to the first harmonic with boundary conditions

$$\Phi_1(\rho, 0, \alpha) = -\frac{R}{1-\nu^2}\sum_{k=1}^{\infty}\cos(2k-1)\alpha\sum_{n=k}^{\infty}\left(\frac{\rho}{c}\right)^{2k+2n-3} a(k+n)\,a(k+n-1) + T(\rho, \alpha)$$

$$|\rho| < \rho(\alpha)$$

$$\Phi_{1z}(\rho, 0, \alpha) = 0, \; |\rho| > \rho(\alpha) \tag{3.1}$$

$$T(\rho, \alpha) = T_1(\alpha)\rho + T_3(\alpha)\rho + T_5(\alpha)\rho + \ldots, \; T_1(\alpha) =$$

$$= \alpha_1^3\cos 3\alpha + \alpha_1^5\cos 5\alpha + \ldots, \; T_3(\alpha) = \alpha_3^5\cos 5\alpha + \alpha_3^7\cos 7\alpha + \ldots, \ldots$$

The pressure under the stamp is connected with the function $\Phi_1(\rho, 0, \alpha)$ by the relation

$$P(\rho, \alpha) = \Phi_{1z}'(\rho, 0, \alpha) \tag{3.2}$$

Problem (3.1) is reduced to the potential problem relative to some function (ξ, η), harmonic in the auxilliary plane $\eta \leqslant 0$ and antisymmetric with respect to ξ. The boundary conditions of this problem are

$$u\eta'(\xi, 0 = 0), \; |\xi| > \rho(\alpha) \; |$$

$$u_\xi'(\xi, 0) = -\frac{R}{1-\nu^2}\sum_{k=1}^{\infty}\cos(2k-1)\alpha\sum_{n=k}^{\infty}\left(\frac{\xi}{c}\right)^{2n-2} + T(\xi, \alpha), \; |\xi| < \rho(\alpha) \tag{3.3}$$

We choose the function $u(\xi, \eta)$ such that the following condition holds:

$$u_\eta'(\xi, 0) = \Phi_{1z}(\xi, 0), \; 0 < \xi < \infty \tag{3.4}$$

4. We will arrive, for every value of α, at mixed problems of potential theory for a half-plane [4]. Solving these problems and using (2.4) and (3.4), we determine $\varphi_z'(\rho, 0)$ and $|\rho| < \rho(\alpha)$:

$$\varphi_z'(\rho, 0) = -\frac{R}{2(1-\nu^2)\sqrt{\rho^2(\alpha) - \rho^2}}\left\{T(\alpha)\left(1 - \frac{\rho}{c}e^{i\alpha}\right)^{-1}\left(1 - \frac{\rho}{c}e^{-i\alpha}\right)^{-1} + \right.$$

$$\left. + Q_0 + \rho Q_1 + (Q_2 + \rho Q_3)(-\rho^2 + {}^1/_2\rho^2(\alpha)) + (Q_4 + \rho Q_5)\left(-\rho^4 + {}^1/_2\rho^2(\alpha)\rho^2 + \frac{1}{8}\rho^4(\alpha)\right) + \ldots\right\}$$

$$T_{(\alpha)} = \left(1 - \left(\frac{\rho(\alpha)}{c}\right)^2\right)^{1/2} \tag{4.1}$$

$$Q_0 = \alpha_0^2\cos 2\alpha + \alpha_0^4\cos 4\alpha + \ldots, \; Q_1 = \alpha_1^3\cos 3\alpha +$$

$$+ \alpha_1^5\cos 5\alpha + \ldots, \; Q_{2n} = \alpha_{2n}^{2n+2}\delta^2\cos(2n+2)\alpha +$$

$$+ \alpha_{2n}^{2n+4}\cos(2n+4)\alpha + \ldots \alpha_{2n}^{2n+2}\cos(2n+4)\alpha + \ldots$$

Here α_n^k are arbitrary constants.

When the stamp is smooth, the function $\varphi_z'(\rho, 0, \alpha)$ can be expanded in the neighbourhood of zero in a Taylor series

TABLE 1.

α	ρ/ρ(α)	P_0	P_1	P_2	P_0	P_1	P_2
0	0	0.724	0.752	0.752	0.721	0.751	0.721
	0.3	1.037	1.100	1.105	1.071	1.074	1.081
	0.6	1.939	2.060	2.087	1.595	1.617	1.620
1.256	0	0.742	0.753	0.753	0.719	0.719	0.719
	0.3	0.845	0.840	0.841	0.803	0.817	0.821
	0.6	1.114	1.084	1.090	1.965	1.003	1.007
2.513	0	0.779	0.753	0.753	0.701	0.710	0.711
	0.3	0.675	0.660	0.655	0.581	0.603	0.613
	0.6	0.709	0.706	0.687	0.537	0.599	0.600
3.781	0	0.779	0.753	0.753	0.709	0.710	0.712
	0.3	0.675	0.660	0.665	0.574	0.591	0.618
	0.6	0.709	0.706	0.687	0.536	0.601	0.623
5.024	0	0.742	0.753	0.753	0.719	0.719	0.720
	0.3	0.845	0.840	0.841	0.818	0.823	0.831
	0.6	0.114	1.084	1.090	0.964	1.002	1.110

$$\varphi_z'(x, y) = A + Bx + Cy \tag{4.2}$$

It therefore follows that the expansion of $\varphi_z'(\rho, \alpha)$ in powers of ρ must contain no terms $\rho^m \cos n\alpha$ when $m<n$:

$$\varphi_z'(\rho, \alpha) = P_0^0 + \rho(P_1^0 + P_1^1 \cos\alpha) + \rho^2 (P_2^0 + P_2^2 \cos 2\alpha) + \dots$$
$$\dots + \rho^{2n}(P_{2n}^0 + P_{2n}^2 \cos 2\alpha + \dots + P_{2n}^{2n} \cos 2n\alpha) + \dots \tag{4.3}$$

The above conditions yield an infinite system of linear algebraic equations for determining the unknown coefficients α_n^k.

We shall solve the problem in question approximately by determining successively $\varphi_z'(\rho, \alpha)$ and using the expansions (4.1) and (4.3) with an accuracy of up to $\rho^2{}_n$ $(n = 0, 1, 2, \dots)$.

In the case when $n = 0 (Q_1, Q_2, Q_3, \dots$ are assumed to be equal to zero), we have

$$\varphi_z'(\rho, \alpha) = \frac{2}{2(1-\nu^2)\sqrt{\rho^2(\alpha)-\rho^2}}\left(T(\alpha)\left(1 - \frac{\rho}{c}e^{i\alpha}\right)^{-1}\left(1 - \frac{\rho}{c}e^{-i\alpha}\right)^{-1} - \right.$$
$$\left. - \sum_{n=1}^{\infty} \alpha^{2n} \cos(2n\alpha)\right), \quad \alpha_0^{2n} = \frac{1}{\pi}\int\left\{-\rho(\alpha)\int T(\alpha)\,d\alpha\left[\int \rho(\alpha)\,d\alpha\right]^{-1} + T(\alpha)\right\}\cos(2\alpha n)\,d\alpha \tag{4.4}$$

Here and henceforth the integration in α is carried out from 0 to 2π.

Similarly, we find for $\varphi_z'(\rho, \alpha)$, when $n = 1$, an expression which differs from (4.4) in having a supplementary term

$$-\frac{R\rho}{2(1-\nu^2)} \sum_{n=1}^{\infty} \alpha_1^{2n+1} \frac{\cos(2n+1)\alpha}{\sqrt{\rho^2(\alpha)-\rho^2}}$$

in its right-hand side.

The unknown constants α_1^{2n+1} are found from the formula

$$\alpha_1^{2n+1} = \frac{1}{\pi}\int\left[T(\alpha)\frac{2\cos\alpha}{c} - \rho(\alpha)(P_1^0 + P_1^1 \cos\alpha)\right]\cos(2n+1)\alpha\,d\alpha$$

$$R = \int \rho(\alpha)\,d\alpha \int \rho(\alpha) \cos^2\alpha\,d\alpha - \left(\int \rho(\alpha)\cos\alpha\,d\alpha\right)^2$$

$$P_1^0 = \frac{1}{R}\left(\int T(\alpha)\frac{2\cos\alpha}{c}d\alpha \int \rho(\alpha)\cos^2\alpha d\alpha - \int T(\alpha)\frac{2\cos^2\alpha}{c}d\alpha \int \rho(\alpha)\cos\alpha\, d\alpha\right)$$

$$P_1^1 = \frac{1}{R}\left(\int \rho(\alpha)\, d\alpha \int T(\alpha)\frac{2\cos^2\alpha}{c}d\alpha - \int \rho(\alpha)\cos\alpha d\alpha \int T(\alpha)\frac{2\cos\alpha}{c}d\alpha\right)$$

Expressions for $\varphi_z'(\rho, \alpha)$ in the case of $n = 2, 3, \ldots$, are obtained in the same manner. As an example, we considered the problem of the pressure exerted by a plane stamp of elliptical and rectangular form in a plane.

The left-hand side of Table 1 gives the values of the pressure $n = 0, 1, 2$, apart from the multiplier $1/2R(1-\nu^2)^{-1}$, in the case of an elliptical stamp (a, b are the semiaxes of the ellipse, $c = \sqrt{[\xi^2+\eta^2}, (\xi, \eta)]$ are the coordinates of the point of application of the force R and $a = 1$, $b = 1, 5$, $c = 3$). On the right-hand side of the table we give the same values for a rectangular stamp in a plane.

REFERENCES

1. GALIN L. A., *Contact Problems of the Theory of Elasticity*. Gostekhizdat, Moscow, 1953.
2. MARTYNENKO M. D., Some three-dimensional contact problems of the theory of elasticity. In *Contact Problems and their Engineering Applications*. NIIMASH, Moscow, 1969.
3. MOSSAKOVSKII V. I., The pressure of a stamp of almost circular cross-section on an elastic half-space. *Prilk. Mat. Mekh.* **18,** 675–680, 1954.
4. MUSKHELISHVILI N. I., *Some Fundamental Problems of the Mathematical Theory of Elasticity*. Nauka, Moscow, 1966.

Translated by L.K.